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THE TORSION OF THE CHARACTERISTIC CONNECTION

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Abstract. In [2], [8], the author studied the characteristic connection as a good substitute for the Levi-Civita connection. In this paper, we consider the space $U(3)/(U(1)\times U(1)\times U(1))$ with an almost Hermitian structure which admits a characteristic connection and compute the characteristic connection concretely.

1. Introduction

Given a G-structure, if the holonomy group with respect to the Levi-Civita connection is the whole group $\mathrm{SO}(n)$, the geometric structure is not preserved by the Levi-Civita connection. But in some situation it is known that there can exist a unique metric connection with skew symmetric torsion which preserves the geometric structure, and it is the characteristic connection. The characteristic connection is a good substitute for the Levi-Civita connection in studying non-integrable geometries([3], [4], [7])).

Furthermore, the characteristic connection and its torsion are very closely related to the string theory in theoretical physics (see [6]).

Recently, many geometric things related to the characteristic connection are studied. For example, in paper [2], the Dirac operator with respect to the characteristic connection was studied.

Unfortunately not every geometric structure admits a characteristic connection. In [8], we considered the homogeneous space $U(3)/(U(1)\times U(1)\times U(1))$ and found an almost Hermitian structure which admits a characteristic connection.

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In this paper, we will see how one can compute the characteristic connection. It is well known that the difference of the characteristic connection, denoted by ∇^{ch}, from the Levi-Civita connection, denoted by ∇^{g}, is the torsion of the characteristic connection ([6]):

$$\nabla^{ch}_X Y = \nabla^g_X Y + \frac{1}{2}T(X,Y).$$

So, we will express the torsion T explicitly.

In section 2, the space $M := U(3)/(U(1)\times U(1)\times U(1))$ is introduced with an almost Hermitian structure (M,g,J) which admits a characteristic connection.

In section 3, we compute the torsion of the characteristic connection ∇^{ch} of (M,g,J).

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2. The homogeneous space $U(3)/(U(1)\times U(1)\times U(1))$

We first introduce a well-known metric family and the Levi-Civita connection of the metrics for a homogeneous reductive space ([1]).

Proposition 2.1. *Let $M = G/H$ be a homogeneous space and β an $\mathrm{Ad}\,(H)$-invariant, positive definite inner product of $\mathfrak{g}$, the Lie algebra of G. For $\mathfrak{m} := \mathfrak{h}^{\perp}$, $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{h}$ is a reductive decomposition. Furthermore we assume that $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2$ with relations*

$$\begin{aligned} &[\mathfrak{h},\mathfrak{m}_1] = \mathfrak{m}_1,\ \ [\mathfrak{m}_1,\mathfrak{m}_1]\subset\mathfrak{h}\oplus\mathfrak{m}_2,\\ &[\mathfrak{h},\mathfrak{m}_2]\subset\mathfrak{m}_2,\ \ [\mathfrak{m}_2,\mathfrak{m}_2]\subset\mathfrak{h},\ \ [\mathfrak{m}_1,\mathfrak{m}_2]\subset\mathfrak{m}_1. \end{aligned}\tag{2.1}$$

Now we consider an $\mathrm{Ad}\,(H)$-invariant inner product on $\mathfrak{m}$ defined by

$$\beta_t := \beta|_{\mathfrak{m}_1\times\mathfrak{m}_1} + 2t\beta|_{\mathfrak{m}_2\times\mathfrak{m}_2},\ \ \textit{for each}\ \ t>0$$

which induces a left invariant metric g_t on G/H.
Then, the Levi-Civita connection of g_t is given by the map $\Lambda_t : \mathfrak{m}\to\mathfrak{so}(\mathfrak{m})$ defined by

$$\begin{aligned} \Lambda_t(X)Y &= \frac{1}{2}[X,Y]_{\mathfrak{m}_2}, &\qquad \Lambda_t(X)B &= t[X,B],\\ \Lambda_t(A)Y &= (1-t)[A,Y], &\qquad \Lambda_t(A)B &= 0, \end{aligned}\tag{2.2}$$

for $X,Y\in\mathfrak{m}_1, A,B\in\mathfrak{m}_2$.

Proof. The map Λ_t, which actually implies the Levi-Civita connection, is uniquely characterized as follows: since the Levi-Civita connection is metric and torsion free (X.2 [9]),

$$\begin{aligned}\Lambda_t(X)Y - \Lambda_t(Y)X &= [X,Y]_{\mathfrak{m}},\\ \beta_t(\Lambda_t(X)Y,Z) + \beta_t(Y,\Lambda_t(X)Z) &= 0.\end{aligned}$$

By direct computaions using (2.1), we can check that the map $\Lambda_t(X)$ definded as (2.2) satisfies the both conditions. □

We now take $G := U(3)$ and $H := U(1) \times U(1) \times U(1) \subset G$ diagonally embedded. Then $M := G/H$ is a 6-dimensional manifold with

$$\mathfrak{g} = \mathfrak{u}(3) = \{A \in M_3(\mathbb{C}) : A + \bar{A}^t = 0\}, \;\; \mathfrak{h} = \{A \in \mathfrak{u}(3) : \;\text{A is diagonal}\}.$$

We define an $\mathrm{Ad}\,(G)$-invariant inner product $\beta := -\frac{1}{2}\mathrm{Re}(\mathrm{tr}AB)$ for $A, B \in \mathfrak{u}(3)$ and decompose $\mathfrak{m} = \mathfrak{h}^{\perp}$ into

$$\mathfrak{m}_1 := \left\{ \begin{bmatrix} 0 & a & b \\ -\bar{a} & 0 & 0 \\ -\bar{b} & 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}, \quad \mathfrak{m}_2 := \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\bar{c} & 0 \end{bmatrix} : c \in \mathbb{C} \right\}.$$

Then we can check that this decomposition satisfies the properties of Proposition 2.1 and we have well defined metrics $g_t, t > 0$.

We use the following notations for basis: Let $D_{kl} = (d_{ij})$ be the $n \times n$ matrix with zero entries except that its (k,l)-entry is 1. Furthermore, let $E_{kl} := D_{kl} - D_{lk}$ for $k \neq l$ and $S_{kl} := i(D_{kl} + D_{lk})$. Then

$$\{e_1 := E_{12},\, e_2 := S_{12},\, e_3 := E_{13},\, e_4 := S_{13},\, e_5 := \frac{1}{\sqrt{2t}}E_{23},\, e_6 := \frac{1}{\sqrt{2t}}S_{23}\}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to β_t. As basis for $\mathfrak{h}$ we take $H_k = S_{kk}/2, k = 1, 2, 3$.

LEMMA 2.2. *(i) The isotropy representation* $\mathrm{Ad} : H \to \mathrm{SO}(6)$ *for* $h = \mathrm{diag}(e^{it}, e^{is}, e^{ir})\,(t, s, r \in \mathbb{R})$ *is given by*

$$\mathrm{Ad}\,(h) = \begin{bmatrix} C(t-s) & 0 & 0 \\ 0 & C(t-r) & 0 \\ 0 & 0 & C(s-r) \end{bmatrix},$$

$$\text{where } C(x) := \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

(ii) The three 2 forms $e_1 \wedge e_2, e_3 \wedge e_4, e_5 \wedge e_6$ *are invariant under the isotropy representation given in (i).*

Proof. See [8]. □

We first consider the Nijeunhuis tensor, $N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$, where $J^2 = -Id$ in an almost complex structure.

LEMMA 2.3. *The Nijeunhuis tensor N satisfies*
1. $N(X,Y) = -N(Y,X)$,
2. $N(X, JY) = -JN(X,Y) = N(JX, Y)$.

Proof. See [8]. □

We now consider a 2-forms and J on G/H, which are well-defined by Lemma 2.2, as follows:

$$\Omega(X,Y) := e_{12} - e_{34} + e_{56} =: g_t(JX, Y) \text{ with } J^2 = -Id, \tag{2.3}$$

where $e_{ij} = e_i \wedge e_j$.

Then by computation,

(2.4)
$$J(e_1) = e_2, J(e_2) = -e_1, J(e_3) = -e_4, J(e_4) = e_3, J(e_5) = e_6, J(e_6) = -e_5.$$

Now we are ready to give the main theorem of [8].

THEOREM 2.1. *On $M = U(3)/(U(1) \times U(1) \times U(1))$ we consider a metric family g_t and an almost complex structure J as above. Then the characteristic connection exists only for $t = \frac{1}{2}$.*

Proof. See [8]. □

3. The torsion of the characteristic connection

Let (M, g) be a manifold with a characteristic connection. We denote the Levi-Civita connection and the characteristic connection by ∇^g and ∇^{ch}, respectively. Then, it is well known that for $X, Y \in TM$ (see [6])

$$\nabla^{ch}_X Y = \nabla^g_X Y + \frac{1}{2} T(X,Y)$$

for some $(2,1)$-tensor T known to be the torsion of the characteristic connection, also called simply as the characteristic torsion. So, it suffices to compute the above torsion T for the characteristic connection ∇^{ch}.

Furthermore, in a Hermitian manifold (M, g, J), the torsion for ∇^{ch} satisfies

$$T(X, Y, -) = (\nabla^g_X J)JY = N(X,Y) + d\Omega(JX, JY, J-). \tag{3.1}$$

Here the $(2,1)$-tensors T, Ω are considered as $(3,0)$-tensors. That is, for T we define $T(X,Y,Z) = g(T(X,Y), Z)$, similarly for Ω.

We now compute the Levi-Civita connection ∇^g using Proposition 2.1. The map $\Lambda_{\frac{1}{2}}$ (see (2.2)) is simply denoted by Λ and we consider E_{ij} with respect to the orthonormal basis e_i of $\mathfrak{m}$. So, E_{ij} actually maps e_i to $-e_j$. The following Lemma is from [1].

Lemma 3.1. *We identify $\mathfrak{m}$ with $\mathbb{R}^6$ and take E_{ij} defined above as basis of $\mathfrak{so}(\mathfrak{m})$. Then*

$$\begin{aligned}
\Lambda(e_1) &= \sqrt{1/4}(E_{35}+E_{46}),\ \Lambda(e_2) = \sqrt{1/4}(E_{45}-E_{36}),\\
\Lambda(e_3) &= \sqrt{1/4}(E_{26}-E_{15}),\ \Lambda(e_4) = -\sqrt{1/4}(E_{16}+E_{25}),\\
\Lambda(e_5) &= \frac{1}{2}(E_{13}+E_{24}),\ \Lambda(e_6) = \frac{1}{2}(E_{14}-E_{23}).
\end{aligned}$$

Proof. First by computations we obtain the following commutators:

$$\begin{aligned}
[e_1,e_2] &= 2(H_1-H_2), & [e_1,e_3] &= -e_5, & [e_1,e_4] &= -e_6,\\
[e_1,e_5] &= e_3, & [e_1,e_6] &= e_4, & [e_2,e_3] &= e_6,\\
[e_2,e_4] &= -e_5, & [e_2,e_5] &= e_4, & [e_2,e_6] &= -e_3,\\
[e_3,e_4] &= 2(H_1-H_3), & [e_3,e_5] &= -e_1, & [e_3,e_6] &= e_2,\\
[e_4,e_5] &= -e_2, & [e_4,e_6] &= -e_1, & [e_5,e_6] &= 2(H_2-H_3).
\end{aligned}$$

We compute $\Lambda(e_1)$. From Proposition 2.1 $\Lambda(e_1)e_i = \frac{1}{2}[e_1,e_i]_{\mathfrak{m}^2}$ for $i = 1,\cdots,4$ and $\Lambda(e_1)e_j = \frac{1}{2}[e_1,e_j]$ for $j = 1,2$. Hence, using the above commutators,

$$\begin{aligned}
\Lambda(e_1)e_1 &= 0\\
\Lambda(e_1)e_2 &= (H_1-H_2)_{\mathfrak{m}^2} = 0,\\
\Lambda(e_1)e_3 &= \frac{1}{2}[e_1,e_3]_{\mathfrak{m}^2} = -\frac{1}{2}e_5,\\
\Lambda(e_1)e_4 &= \frac{1}{2}[e_1,e_4]_{\mathfrak{m}^2} = -\frac{1}{2}e_6,\\
\Lambda(e_1)e_5 &= \frac{1}{2}[e_1,e_5] = \frac{1}{2}e_3,\\
\Lambda(e_1)e_6 &= \frac{1}{2}[e_1,e_6] = \frac{1}{2}e_4.
\end{aligned}$$

We consider E_{ij} which maps e_i of $\mathfrak{m}$ and e_j to e_i. Then we have

$$\Lambda(e_1) = \frac{1}{2}(E_{35}+E_{46}).$$

We obtain the other results by similar computations. □

Lemma 3.2.

$$\Lambda_t(e_i)w = \sum_j (e_j \lrcorner \Lambda_t(e_i)) \wedge (e_j \lrcorner w), \tag{3.2}$$

Proof. An element of the Lie algebra $\mathfrak{so}(n)$ acts on $\Lambda^2\mathbb{R}^6$ as follows: let $A \in \mathfrak{so}(n)$ and $e_i, e_j \in \mathbb{R}^6$ then

$$A(e_i \wedge e_j) := A(e_i) \wedge e_j + e_i \wedge A(e_j).$$

$\Lambda_t(e_i)$ is an element of the Lie algebra $\mathfrak{so}(\mathfrak{m})$ and $\mathfrak{m}$ can be identified with $\mathbb{R}^6$. And $\Lambda_t(e_i)$ is a linear combination of E_{ij}'s (Lemma 3.1). So we investigate the action of E_{ij} on $\Lambda^2\mathbb{R}^6$. For $e_k \wedge e_l \in \Lambda^2\mathbb{R}^6, k \neq l$,

$$\begin{aligned} E_{ij}(e_k \wedge e_l) &= E_{ij}(e_k) \wedge e_l + e_k \wedge E_{ij}(e_l) \\ &= (-\delta_{ik} e_j + \delta_{jk} e_i) \wedge e_l + e_k \wedge (-\delta_{il} e_j + \delta_{jl} e_i) \\ &= -e_j \wedge e_i \\ &= (e_i \lrcorner E_{ij}) \wedge (e_i \lrcorner (e_i \wedge e_l)). \qquad (3.3) \end{aligned}$$

(3.3) holds for all the basis elements of $\mathfrak{m}$ and it implies the formula (3.2). □

Let $M = U(3)/(U(1) \times U(1) \times U(1))$ with metric $g_{\frac{1}{2}}$ and an almost complex structure J as follows:

$$J(e_1) = e_2, J(e_2) = -e_1, J(e_3) = -e_4, J(e_4) = e_3, J(e_5) = e_6, J(e_6) = -e_5.$$

Note that the above J is induced from a 2-form $\Omega(X,Y) := e_{12} - e_{34} + e_{56} =: g_{\frac{1}{2}}(JX, Y)$. Then we obtain:

Theorem 3.3. *The manifold* $(M, g_{\frac{1}{2}}, J)$ *as above admits a characteristic connection*

$$\nabla^{ch}_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y, -),$$

with $T = e_{245} - e_{236} + e_{135} + e_{146}$, *where* e_{ijk} *means* $e_i \wedge e_j \wedge e_k$.

Proof. By (3.1), we need to compute $d\Omega$ and N in (M, g, J).

i) First $d\Omega$ is given by

$$d\Omega = \sum_i e_i \wedge \nabla^g_{e_i} \Omega,$$

so we compute $\nabla^g_{e_i}\Omega$, $\Omega = e_{12} - e_{34} + e_{56}$, $i = 1, \cdots, 6$. Recall that the three 2-forms $w =e_{12}$, e_{34}, e_{56} are invariant under the isotropy representation (Lemma 2.2). And from Proposition 2.1 and Lemma 3.2,

$$\nabla^g_{e_i} w = \Lambda_t(e_i)w = \sum_j (e_j \lrcorner \Lambda_t(e_i)) \wedge (e_j \lrcorner w). \tag{3.4}$$

Note that E_{ij} maps e_i to $-e_j$, so E_{ij} can be identified with the two form $-e_{ij}$.

From Lemma 3.1 $\Lambda(e_1) = \frac{1}{2}(E_{35} + E_{46})$ identified with $\frac{1}{2}(e_{35} + e_{46})$, so $e_j \lrcorner \Lambda(e_1) = 0$ for $j = 1, 2$ and (3.4) implies

$$\nabla_{e_1} e_{12} = \nabla_{e_2} e_{12} = 0.$$

Similarly

$$\nabla_{e_3} e_{34} = \nabla_{e_4} e_{34} = 0$$

and

$$\nabla_{e_5} e_{56} = \nabla_{e_6} e_{56} = 0.$$

Now

$$\begin{aligned}
\nabla_{e_1} e_{34} &= \sum_j (e_j \lrcorner \Lambda(e_1)) \wedge (e_j \lrcorner e_{34}) \\
&= -\frac{1}{2} \sum_j (e_j \lrcorner (e_{35} + e_{46})) \wedge (e_j \lrcorner e_{34}) \\
&= -\frac{1}{2} \left((e_3 \lrcorner (e_{35} + e_{46})) \wedge (e_3 \lrcorner e_{34}) + (e_4 \lrcorner (e_{35} + e_{46})) \wedge (e_4 \lrcorner e_{34})\right) \\
&= -\frac{1}{2}(e_5 \wedge e_4 - e_6 \wedge e_3) \\
&= \frac{1}{2}(e_{45} - e_{36})
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{e_1} e_{56} &= \sum_j (e_j \lrcorner \Lambda(e_1)) \wedge (e_j \lrcorner e_{56}) \\
&= -\frac{1}{2} \sum_j (e_j \lrcorner (e_{35} + e_{46})) \wedge (e_j \lrcorner e_{56}) \\
&= -\frac{1}{2} \left((e_5 \lrcorner (e_{35} + e_{46})) \wedge (e_5 \lrcorner e_{56}) + (e_6 \lrcorner (e_{35} + e_{46})) \wedge (e_6 \lrcorner e_{56})\right) \\
&= -\frac{1}{2}(-e_3 \wedge e_6 + e_4 \wedge e_5) \\
&= \frac{1}{2}(e_{36} - e_{45}).
\end{aligned}$$

Similarly,

$$\nabla_{e_2} e_{34} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{45} - e_{36})) \wedge (e_j \lrcorner e_{34}) = -\frac{1}{2}(e_{46} + e_{35}).$$

$$\nabla_{e_2} e_{56} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{45} - e_{36})) \wedge (e_j \lrcorner e_{56}) = \frac{1}{2}(e_{46} + e_{35}).$$

$$\nabla_{e_3} e_{12} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{26} - e_{15})) \wedge (e_j \lrcorner e_{12}) = -\frac{1}{2}(e_{16} + e_{25}).$$

$$\nabla_{e_3} e_{56} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{26} - e_{15})) \wedge (e_j \lrcorner e_{56}) = -\frac{1}{2}(e_{16} + e_{25}).$$

$$\nabla_{e_4} e_{12} = \frac{1}{2}\sum_j (e_j \lrcorner (e_{16} + e_{25})) \wedge (e_j \lrcorner e_{12}) = \frac{1}{2}(e_{15} - e_{26}).$$

$$\nabla_{e_4} e_{56} = \frac{1}{2}\sum_j (e_j \lrcorner (e_{16} + e_{25})) \wedge (e_j \lrcorner e_{56}) = \frac{1}{2}(e_{15} - e_{26}).$$

$$\nabla_{e_5} e_{12} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{13} + e_{24})) \wedge (e_j \lrcorner e_{12}) = \frac{1}{2}(e_{23} - e_{14}).$$

$$\nabla_{e_5} e_{34} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{13} + e_{24})) \wedge (e_j \lrcorner e_{34}) = \frac{1}{2}(e_{14} - e_{23}).$$

$$\nabla_{e_6} e_{12} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{14} - e_{23})) \wedge (e_j \lrcorner e_{12}) = \frac{1}{2}(e_{13} + e_{24}).$$

$$\nabla_{e_6} e_{34} = -\frac{1}{2}\sum_j (e_j \lrcorner (e_{14} - e_{23})) \wedge (e_j \lrcorner e_{34}) = -\frac{1}{2}(e_{13} + e_{24}).$$

Note that

$$\begin{aligned} \nabla_{e_1} e_{34} + \nabla_{e_1} e_{56} &= \nabla_{e_2} e_{34} + \nabla_{e_2} e_{56} = \nabla_{e_5} e_{12} + \nabla_{e_5} e_{34} \\ &= \nabla_{e_6} e_{12} + \nabla_{e_6} e_{34} = 0, \\ \nabla_{e_3} e_{12} &= \nabla_{e_3} e_{56}, \quad \nabla_{e_4} e_{12} = \nabla_{e_4} e_{56}. \end{aligned}$$

So, we have

$$\begin{aligned} d\Omega &= \sum_i e_i \wedge \nabla^g_{e_i}\Omega \\ &= \sum_i e_i \wedge \nabla^g_{e_i}(e_{12} - e_{34} + e_{56}) \\ &= 2(e_1 \wedge \nabla^g_{e_1} e_{56} + e_2 \wedge \nabla^g_{e_2} e_{56} + e_3 \wedge \nabla_{e_3} e_{12} + e_4 \wedge \nabla_{e_4} e_{12} \\ &\quad + \ e_5 \wedge \nabla_{e_5} e_{12} + e_6 \wedge \nabla_{e_6} e_{12}) \\ &= e_1 \wedge (e_{36} - e_{45}) + e_2 \wedge (e_{46} + e_{35}) - e_3 \wedge (e_{16} + e_{25}) \\ &\quad + e_4 \wedge (e_{15} - e_{26}) + e_5 \wedge (e_{23} - e_{14}) + e_6 \wedge (e_{13} + e_{24}) \\ &= 3(e_{136} - e_{145} + e_{246} + e_{235}). \end{aligned}$$

And from (2.4)

$$d\Omega(J) = -3(e_{245} - e_{236} + e_{135} + e_{146}).$$

ii) The Nijenhuis tensor N.

Using Lemma 2.3, by computation we have (for details see [8])

$$N(e_1, e_2) = N(e_3, e_4) = N(e_5, e_6) = 0.$$

And

$$\begin{aligned} N(e_1, e_3) &= N(e_2, e_4) = 4e_5, \\ N(e_1, e_5) &= -N(e_2, e_6) = -4e_3, \\ N(e_3, e_5) &= N(e_4, e_6) = 4e_1, \end{aligned}$$

$$\begin{aligned} N(e_1, e_4) &= N(e_2, e_3) = -4e_6, \\ N(e_1, e_6) &= N(e_2, e_5) = -4e_4, \\ N(e_3, e_6) &= N(e_4, e_5) = 4e_2. \end{aligned}$$

So as a $(3, 0)$-tensor,

$$N = 4(e_{245} - e_{236} + e_{135} + e_{146}).$$

iii) By **i)**, **ii)** and (3.1)

$$T = N + d\Omega(J) = e_{245} - e_{236} + e_{135} + e_{146}.$$

□

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